Improper Integrals

Example

Compute $\int_1^\infty x^{-2}\,dx$

$$\begin{aligned}\int_1^\infty x^{-2}\,dx &= \lim_{k\to\infty}\int_1^k x^{-2}\,dx\\ &= \lim_{k\to\infty}\left[-x^{-1}\right]_1^k\\ &= \lim_{k\to\infty}\left(1-\frac{1}{k}\right)\\ &\to 1\end{aligned}$$

Example

Compute $\int_1^\infty x^{-1}\,dx$

$$\begin{aligned}\int_1^\infty x^{-1}\,dx &= \lim_{k\to\infty}\int_1^k x^{-1}\,dx\\ &= \lim_{k\to\infty}\left[\ln x\right]_1^k\\ &\to \infty\end{aligned}$$

Example

Compute $\int_0^\infty e^{-3x}\,dx$

$$\begin{aligned}\int_0^\infty e^{-3x}\,dx &= \lim_{k\to\infty}\int_0^k e^{-3x}\,dx\\ &= \lim_{k\to\infty}\left[-\frac{1}{3}e^{-3x}\right]_0^k\\ &= \lim_{k\to\infty}\left(-\frac{1}{3}e^{-3k}+\frac{1}{3}\right)\\ &= \frac{1}{3}\end{aligned}$$

Example

Compute $\int_0^5 \frac{1}{\sqrt[3]{x-3}}dx$

$$
\begin{aligned}
\int_0^5 \frac{1}{\sqrt[3]{x-3}}dx &= \lim_{c\nearrow 3}\int_0^c \frac{1}{\sqrt[3]{x-3}}dx + \lim_{d\searrow 3}\int_d^5 \frac{1}{\sqrt[3]{x-3}}dx \\
\lim_{c\nearrow 3}\int_0^c \frac{1}{\sqrt[3]{x-3}}dx &= \lim_{c\nearrow 3}\left[\frac{3}{2}(x-3)^{2/3}\right]_0^c \\
&= \lim_{c\nearrow 3}\left(\frac{3}{2}(c-3)^{2/3} - \frac{3}{2}(-3)^{2/3}\right) \\
&= -\frac{3}{2}(-3)^{2/3} \\
\lim_{d\searrow 3}\int_d^5 \frac{1}{\sqrt[3]{x-3}}dx &= \lim_{d\searrow 3}\left[\frac{3}{2}(x-3)^{2/3}\right]_d^5 \\
&= \lim_{d\searrow 3}\left(\frac{3}{2}(2)^{2/3} - \frac{3}{2}(d-3)^{2/3}\right) \\
&= \frac{3}{2}(2)^{2/3} \\
\Rightarrow \qquad \int_0^5 \frac{1}{\sqrt[3]{x-3}}dx &= \frac{3}{2}\left(\sqrt[3]{4} - \sqrt[3]{9}\right)
\end{aligned}
$$

Example

Compute $\int_{-\infty}^{\infty} x\,dx$

$$
\begin{aligned}
\int_{-\infty}^{\infty} x\,dx &= \lim_{k\to\infty}\int_0^k x dx + \lim_{K\to-\infty}\int_K^0 x dx \\
&= \lim_{k\to\infty}\frac{1}{2}k^2 - \lim_{K\to\infty}\frac{1}{2}K^2
\end{aligned}
$$

Partial Fractions

Example

Write $\frac{1}{(x+5)(x+4)}$ as partial fractions

$$
\begin{aligned}
& \frac{1}{(x+5)(x+4)} = \frac{A}{x+5} + \frac{B}{x+4} \\
\Rightarrow\quad & 1 = A(x+4) + B(x+5) \\
x=-4: \quad & 1 = A((-4)+4) + B((-4)+5) \\
& \quad = 0 \cdot A + B \\
\Rightarrow\quad & B = 1 \\
x=-5: \quad & 1 = A((-5)+4) + B((-5)+5) \\
& \quad = -A + 0 \cdot B \\
\Rightarrow\quad & A = -1 \\
\Rightarrow\quad & \frac{1}{(x+5)(x+4)} = \frac{1}{x+4} - \frac{1}{x+5}
\end{aligned}
$$

Example

Write $\frac{1}{(x+2)(x-1)^2}$ as partial fractions

$$
\begin{aligned}
& \frac{1}{(x+2)(x-1)^2} = \frac{A}{x+2} + \frac{Bx+C}{(x-1)^2} \\
& \quad = \frac{A}{x+2} + \frac{B}{x-1} + \frac{C'}{(x-1)^2} \\
\Rightarrow\quad & 1 = A(x-1)^2 + B(x-1)(x+2) + C'(x+2) \\
x=-2: \quad & 1 = A((-2)-1)^2 + B((-2)-1)((-2)+2) + C'((-2)+2) \\
& \quad = 9A \\
\Rightarrow\quad & A = \frac{1}{9} \\
x=1: \quad & 1 = A(1-1)^2 + B(1-1)(1+2) + C'(1+2) \\
\Rightarrow\quad & C' = \frac{1}{3} \\
x=0: \quad & 1 = A(-1)^2 + B(-1)(2) + C'(2) \\
\Rightarrow\quad & \quad = \frac{1}{9} - 2B + \frac{2}{3} \\
\Rightarrow\quad & B = -\frac{1}{9} \\
\Rightarrow\quad & \frac{1}{(x+2)(x-1)^2} = \frac{1}{9(x+2)} - \frac{1}{9(x-1)} + \frac{1}{3(x-1)^2}
\end{aligned}
$$

Example

Write $\dfrac{3x^2-3x-2}{(x-1)(x-2)}$ as partial fractions

$$\begin{aligned}
 & \frac{3x^2-3x-2}{(x-1)(x-2)} = \frac{3(x-1)(x-2)+6x-8}{(x-1)(x-2)} \\
 & \qquad = 3 + \frac{6x-8}{(x-1)(x-2)} \\
\textit{Suppose} \quad & \frac{6x-8}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2} \\
\Rightarrow \quad & 6x-8 = A(x-2)+B(x-1) \\
x=2: \quad & 4 = B \\
x=1: \quad & -2 = -A \\
\Rightarrow \quad & \frac{3x^2-3x-2}{(x-1)(x-2)} = 3 + \frac{2}{x-1} + \frac{4}{x-2}
\end{aligned}$$

Example

Write $\dfrac{x^2+5x+7}{(x+1)^3}$ as partial fractions

$$\begin{aligned}
\frac{x^2+5x+7}{(x+1)^3} &= \frac{(x+1)^2+3x+7}{(x+1)^3} \\
&= \frac{1}{x+1} + \frac{3x+7}{(x+1)^3} \\
&= \frac{1}{x+1} + \frac{3(x+1)+4}{(x+1)^3} \\
&= \frac{1}{x+1} + \frac{3}{(x+1)^2} + \frac{4}{(x+1)^3}
\end{aligned}$$

Example

Write $\dfrac{1}{x^2-2x-1}$ as partial fractions

$$\begin{aligned}
&& \frac{1}{x^2-2x-1} &= \frac{1}{\left(x-(1+\sqrt{2})\right)\left(x-(1-\sqrt{2})\right)} \\
&& &= \frac{A}{x-(1+\sqrt{2})} + \frac{B}{x-(1-\sqrt{2})} \\
\Rightarrow && 1 &= A\left(x-(1-\sqrt{2})\right) + B\left(x-(1+\sqrt{2})\right) \\
x = 1-\sqrt{2}: && 1 &= B((1-\sqrt{2})-(1+\sqrt{2})) \\
\Rightarrow && 1 &= B(-2\sqrt{2}) \\
\Rightarrow && B &= -\frac{\sqrt{2}}{4} \\
x = 1+\sqrt{2}: && 1 &= A((1+\sqrt{2}-(1-\sqrt{2})) \\
\Rightarrow && 1 &= A(2\sqrt{2}) \\
\Rightarrow && A &= \frac{\sqrt{2}}{4} \\
\Rightarrow && \frac{1}{x^2-2x-1} &= \frac{\sqrt{2}}{4}\left(\frac{1}{x-(1+\sqrt{2})} - \frac{1}{x-(1-\sqrt{2})}\right)
\end{aligned}$$

Example

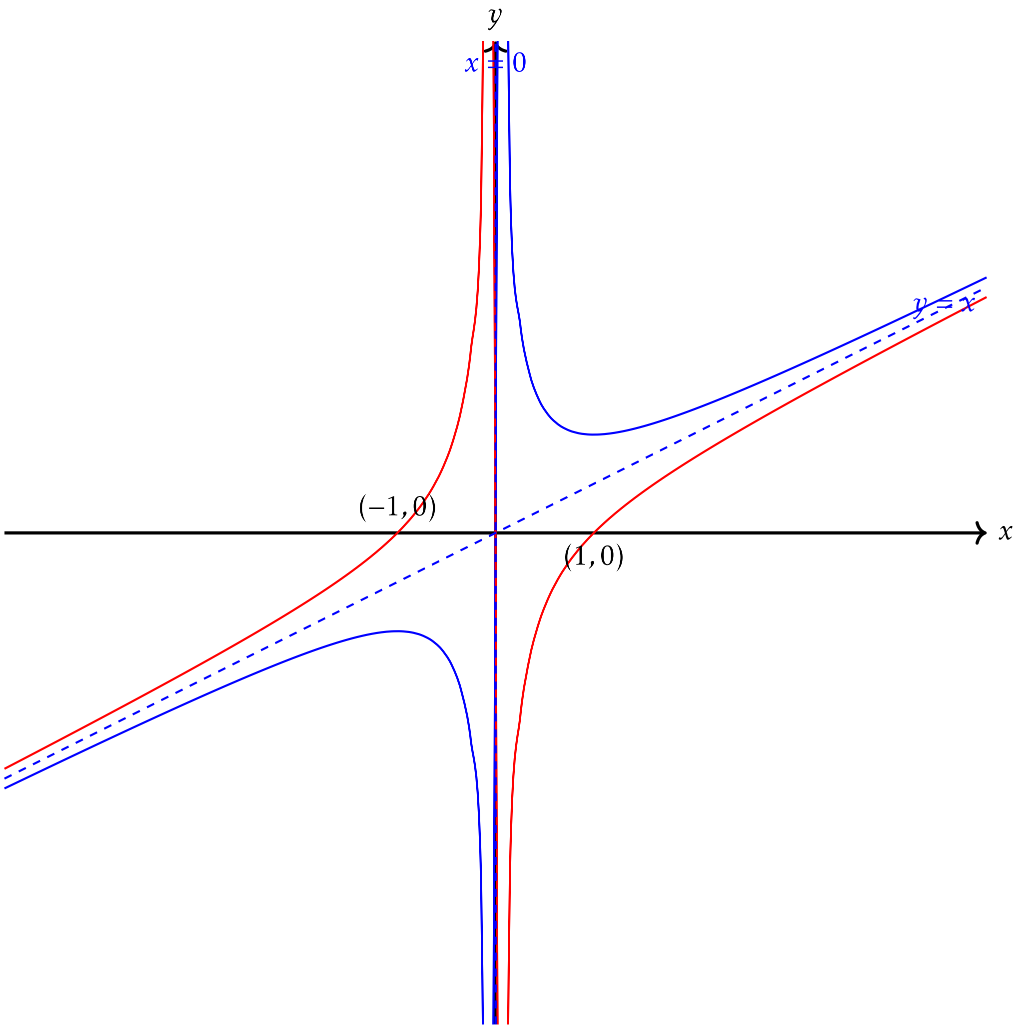
Write $\dfrac{1}{x^3-1}$ as partial fractions

$$\begin{aligned}
&& \frac{1}{x^3-1} &= \frac{1}{(x-1)(x^2+x+1)} \\
&& &= \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} \\
\Rightarrow && 1 &= A(x^2+x+1) + (Bx+C)(x-1) \\
x = 1: && 1 &= 3A \\
\Rightarrow && A &= \frac{1}{3} \\
x = 0: && 1 &= \frac{1}{3} - C \\
\Rightarrow && C &= -\frac{2}{3} \\
x = -1: && 1 &= A + (-B+C)(-2) \\
\Rightarrow && 1 &= \frac{1}{3} + 2B + \frac{4}{3} \\
\Rightarrow && B &= -\frac{1}{3} \\
\Rightarrow && \frac{1}{x^3-1} &= \frac{1}{3(x-1)} - \frac{x+2}{3(x^2+x+1)}
\end{aligned}$$

Graph Sketching of Rational Functions

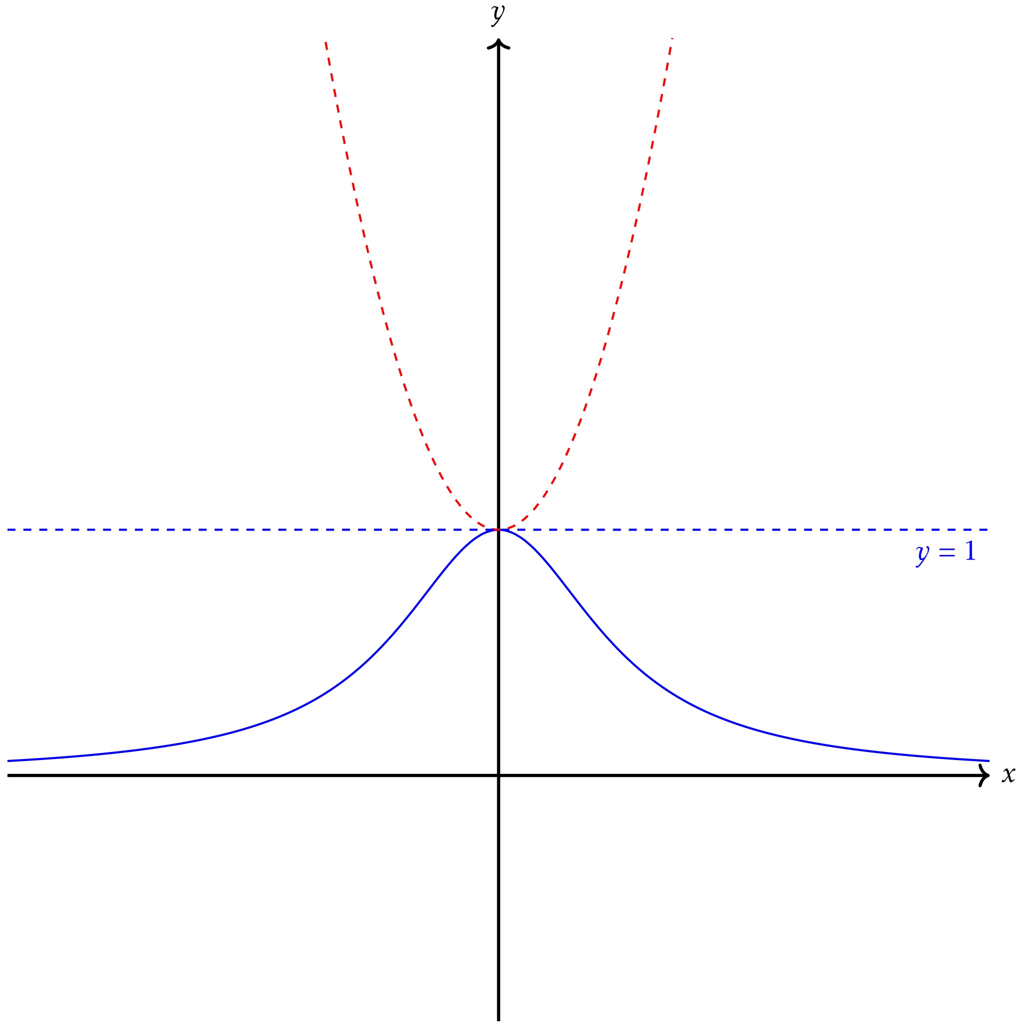
Example

Sketch $y = x + \frac{1}{x}$ and $y = x - \frac{1}{x}$



Example

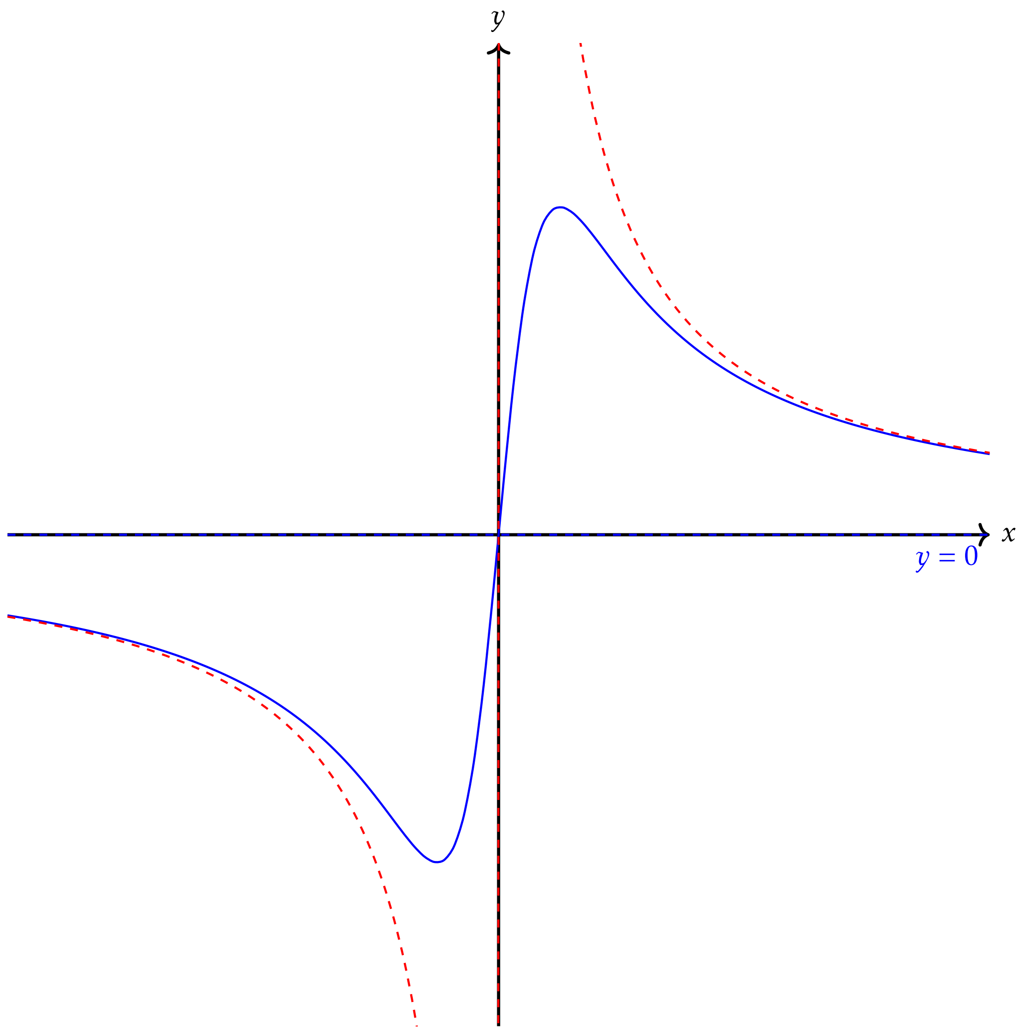
Sketch $y = \dfrac{1}{1+x^2}$



1. *What are the turning points on this curve?*
2. *What are the points of inflection? How do we know they have them? [Without calcuation!]*

Example

Sketch $y = \dfrac{2x}{1+x^2}$

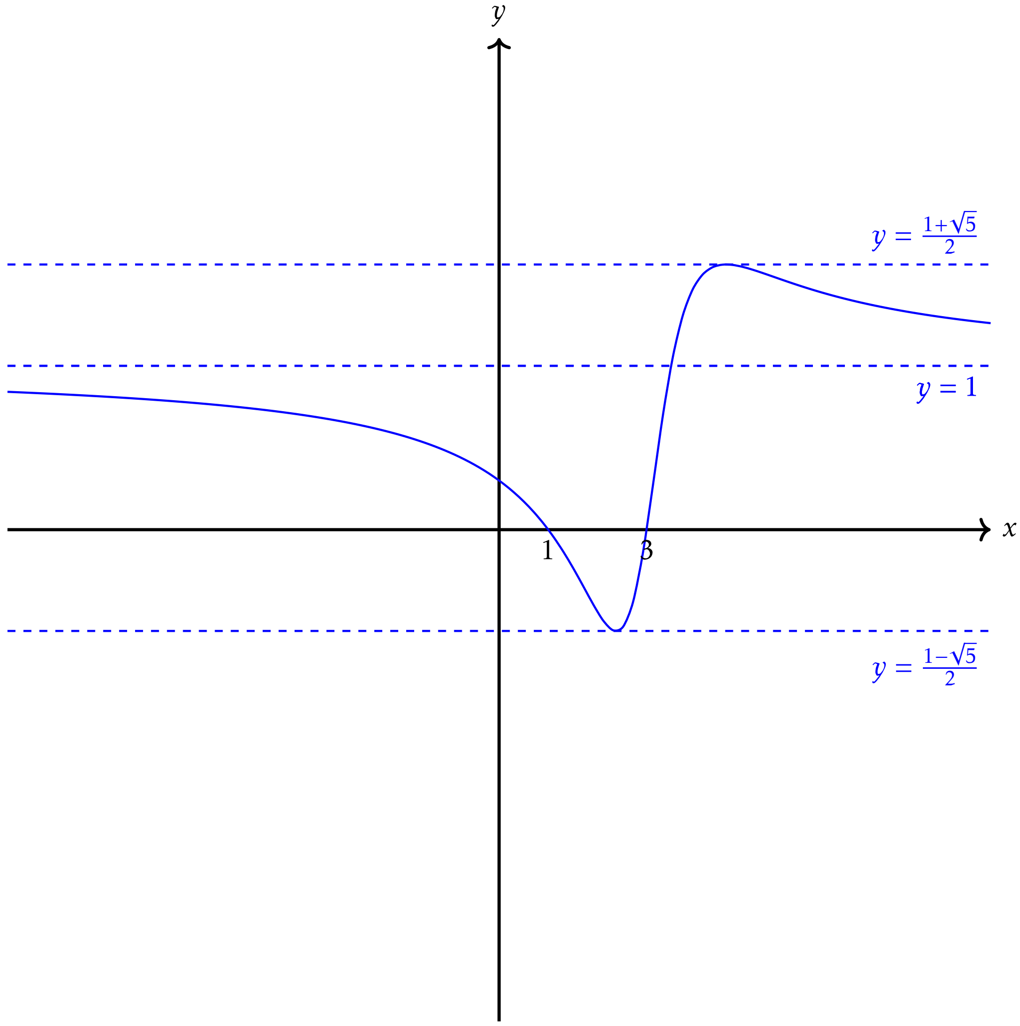


1. *What are the turning points on this curve?*
2. *What are the points of inflection? How do we know they have them? [Without calcuation!]*

Example

Sketch $y = \dfrac{x^2 - 4x + 3}{x^2 - 6x + 10}$

Notice that $x^2 - 4x + 3 = (x-3)(x-1)$ *and the denominator is always positive.* $(x-3)^2 + 1$.
Also recall from back when we were studying inequalities, $y \in \left[\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right]$



Uses of Partial Fractions

Integration

Example

Integrate $\displaystyle\int_2^3 \frac{x+1}{(x-1)(x^2+1)}\,dx$

$$
\begin{aligned}
& & \frac{x+1}{(x-1)(x^2+1)} &= \frac{A}{x-1} + \frac{Bx+C}{x^2+1} \\
&\Rightarrow & x+1 &= A(x^2+1) + (Bx+C)(x-1) \\
&x=1: & 2 &= 2A \\
&\Rightarrow & A &= 1 \\
&x=0: & 1 &= A + C(-1) \\
&\Rightarrow & C &= 0 \\
&\Rightarrow & x+1 &= x^2+1+Bx(x-1) \\
&\Rightarrow & B &= -1
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \quad \int_2^3 \frac{x+1}{(x-1)(x^2+1)}\,dx &= \int_2^3 \left(\frac{1}{x-1} - \frac{x}{x^2+1}\right) dx \\
&= \left[\ln(x-1) - \frac{1}{2}\ln(x^2+1)\right]_2^3 \\
&= \ln 2 - \frac{1}{2}\ln(10) + \frac{1}{2}\ln(5) \\
&= \ln 2 - \frac{1}{2}\ln 2 \\
&= \frac{1}{2}\ln 2
\end{aligned}
$$

Example

Integrate $\displaystyle\int_{-2}^2 \frac{x^2+x}{(x-4)(x^2+4)}\,dx$

$$
\begin{aligned}
& & \frac{x^2+x}{(x-4)(x^2+4)} &= \frac{A}{x-4} + \frac{Bx+C}{x^2+4} \\
&\Rightarrow & x^2+x &= A(x^2+4) + (Bx+C)(x-4) \\
&x=4: & 20 &= 20A \\
&\Rightarrow & A &= 1 \\
&x=0: & 0 &= 4-4C \\
&\Rightarrow & C &= 1
\end{aligned}
$$

$$
\begin{aligned}
x = 1: \quad & 2 = 5A + (B + C)(-3) \\
\Rightarrow \quad & B = 0
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \quad \int_{2}^{2} \frac{x^2 + x}{(x-4)(x^2+4)}\, dx &= \int_{-2}^{2} \left(\frac{1}{x-4} + \frac{1}{x^2+4} \right) dx \\
&= \left[\ln|x-4| + \frac{1}{2}\tan^{-1}\frac{x}{2} \right]_{-2}^{2} \\
&= \ln(2) - \ln(6) + \frac{1}{2}\tan^{-1} 1 - \frac{1}{2}\tan^{-1}(-1) \\
&= -\ln 3 + \frac{\pi}{4}
\end{aligned}
$$

Differentiation

Example

Write $\dfrac{3x^2-3x-2}{(x-1)(x-2)}$ as partial fractions and hence differentiate it.

$$\begin{aligned}
&& \frac{3x^2-3x-2}{(x-1)(x-2)} &= \frac{3(x-1)(x-2)+6x-8}{(x-1)(x-2)} \\
&& &= 3+\frac{6x-8}{(x-1)(x-2)} \\
\textit{Suppose} && \frac{6x-8}{(x-1)(x-2)} &= \frac{A}{x-1}+\frac{B}{x-2} \\
\Rightarrow && 6x-8 &= A(x-2)+B(x-1) \\
x=2: && 4 &= B \\
x=1: && -2 &= -A \\
\Rightarrow && \frac{3x^2-3x-2}{(x-1)(x-2)} &= 3+\frac{2}{x-1}+\frac{4}{x-2} \\
\Rightarrow && \frac{d}{dx}\left(\frac{3x^2-3x-2}{(x-1)(x-2)}\right) &= \frac{d}{dx}\left(3+\frac{2}{x-1}+\frac{4}{x-2}\right) \\
&& &= 0-\frac{2}{(x-1)^2}-\frac{4}{(x-2)^2}
\end{aligned}$$

Example

Split $\dfrac{x^4}{(x^2+4)(x-2)^2}$ into partial fractions and hence differentiate it

$$\begin{aligned}
&& \frac{x^4}{(x^2+4)(x-2)^2} &= 1+\frac{A}{x-2}+\frac{B}{(x-2)^2}+\frac{Cx+D}{x^2+4} \\
\Rightarrow && x^4 &= (x^2+4)(x-2)^2+A(x-2)(x^2+4)+B(x^2+4)+(Cx+D)(x-2)^2 \\
x=2: && 16 &= 8B \\
\Rightarrow && B &= 2 \\
x=0: && 0 &= 16-8A+4B+4D \\
\Rightarrow && 2A-D &= 6 \\
x=1: && 1 &= 5-5A+5B+C+D \\
\Rightarrow && 5A-C-D &= 14 \\
x=-1: && 1 &= 45-15A+5B+9(C-D) \\
\Rightarrow && 15A-9C+9D &= 54 \\
\Rightarrow && A,B,C,D &= 3,2,1,0 \\
\Rightarrow && \frac{d}{dx}\left(\frac{x^4}{(x^2+4)(x-2)^2}\right) &= \frac{d}{dx}\left(1+\frac{3}{x-2}+\frac{2}{(x-2)^2}+\frac{x}{x^2+4}\right)
\end{aligned}$$

$$= 0 - \frac{3}{(x-2)^2} - \frac{4}{(x-2)^3} + \frac{4-x^2}{(x^2+4)^2}$$

Binomial Expansion

Example

Find the Binomial expansion of $\dfrac{5x-4}{(x+2)(x^2+3)}$ up to and including the term in x^2

$$
\begin{aligned}
& & \frac{5x-4}{(x+2)(x^2+3)} &= \frac{A}{x+2} + \frac{Bx+C}{x^2+3} \\
&\Rightarrow & 5x-4 &= A(x^2+3) + (Bx+C)(x+2) \\
&x=-2: & -14 &= 7A \\
&\Rightarrow & A &= -2 \\
&x=0: & -4 &= -6+2C \\
&\Rightarrow & C &= 1 \\
&x=1: & 1 &= (-2)(4) + (B+1)3 \\
&\Rightarrow & B &= 2 \\
&\Rightarrow & \frac{5x-4}{(x+2)(x^2+3)} &= \frac{-2}{x+2} + \frac{2x+1}{x^2+3} \\
& & &= \frac{-1}{1+\frac{1}{2}x} + \frac{2x+1}{3(1+\frac{1}{3}x^2)} \\
& & &= -1(1+\tfrac{1}{2}x)^{-1} + \tfrac{1}{3}(2x+1)(1+\tfrac{1}{3}x^2)^{-1} \\
& & &= -1(1-\tfrac{1}{2}x+(\tfrac{1}{2}x)^2+\cdots) + \tfrac{1}{3}(2x+1)(1-\tfrac{1}{3}x^2+\cdots) \\
& & &= -1+\tfrac{1}{2}x-\tfrac{1}{4}x^2+\tfrac{1}{3}-\tfrac{1}{9}x^2+\tfrac{2}{3}x+\cdots \\
& & &= -\tfrac{2}{3}+\tfrac{7}{6}x-\tfrac{13}{36}x^2+\cdots
\end{aligned}
$$